

A lattice worldsheet sum for 4-d Euclidean general relativity

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Abstract

A lattice model for four dimensional Euclidean quantum general relativity is proposed for a simplicial spacetime. It is shown how this model can be expressed in terms of a sum over worldsheets of spin networks in the lattice, and an interpretation of these worldsheets as spacetime geometries is given, based on the geometry defined by spin networks in canonical loop quantized GR. The spacetime geometry has a Planck scale discreteness which arises "naturally" from the discrete spectrum of spins of $SU(2)$ representations (and not from the use of a spacetime lattice).

The lattice model of the dynamics is a formal quantization of the classical lattice model of [Rei97a], which reproduces, in a continuum limit, Euclidean general relativity.

1 Introduction

The present work aims to provide a step in the construction of a theory of "quantum general relativity" (QGR), meaning a theory within the framework of standard quantum mechanics¹ the classical limit of which is general

¹Except, perhaps, in the role played by time.

relativity (GR), and in which the four dimensional diffeomorphism invariance of GR is realized exactly, without quantum anomalies.

This conservative approach to the quantum gravity, in which one attempts to quantize GR without modification at the classical level or unification with other fields, has been revived by Ashtekar's discovery [Ash86] [Ash87] of convenient new variables for classical canonical GR, and the application of loop quantization [GT86] to Ashtekar's canonical theory by Rovelli and Smolin [RS88].

In Ashtekar's canonical theory the canonical variables are the left-handed (self-dual)² part of the spin connection on 3-space and, conjugate to it, the densitized dreibein. The connection can thus be taken as the configuration variables, opening the door to a loop quantization of GR.

In loop quantization one supposes that the state can be represented by a power series³ in the spatial Wilson loops of the connection (which coordinate the connections up to gauge), so the fundamental excitations are loops created by the Wilson loop operators.

At the present time the kinematics of loop quantized canonical GR is fairly well understood (see [ALMMT95] for a recent review). That is to say, the space of states invariant under 3-diffeomorphisms of 3-space has been identified. The kinematics alone leads to the striking prediction that geometrical observables measuring lengths [Thi96f], areas [RS95][AL96a] [FLR96], and volumes [RS95],[AL96b] have discrete spectra and finite, Planck scale, lowest non-zero eigenvalues.

However, the dynamics of the theory, is not well understood. The dynamics of QGR is encoded in the restrictions placed on physical states by the requirement of full *4-diffeo* invariance. These restrictions are represented in the classical theory by the scalar, or "Hamiltonian", constraint (which, when formally quantized yields the Wheeler-deWitt equation). ⁴

²In euclidean GR the frame rotation group is $SO(4)$ which can be written as the product $SU(2)_R \otimes SU(2)_L$. Left handed tensors transform only under the $SU(2)_L$ factor. Examples are left handed spinors and self-dual antisymmetric tensors, i.e. tensors a that satisfy $a^{[IJ]} = \epsilon^{IJ}{}_{KL}a^{KL}$.

³The series is not assumed to be convergent. A divergent series still defines a distribution on the space of generalized connections via the Ashtekar-Lewandowski [ALMMT95] inner product.

⁴In fact since the loop quantization of GR remains incomplete in the sense that no satisfactory quantum dynamics has been found, it remains possible that no such quantum dynamics even *can* be accomodated within the loop kinematics. If this is the case the

Thiemann [Thi96b] has recently proposed a rigorously defined scalar constraint operator in loop quantized GR. However, it seems that the theory defined by this constraint does not have GR as its classical limit [Smo96], [GLMP97].

In a theory defined in terms of path integrals over 4-diffeo equivalence classes of histories 4-diffeo invariance is incorporated from the outset. Can quantum GR be set up within the kinematical framework of loop quantization with transition amplitudes defined by sums over 4-diffeo equivalence classes of histories? In addition to manifest 4-diffeo invariance such a theory would incorporate the Planck scale discreteness of geometry implied by the loop kinematics, which might provide a physical UV cutoff, both for the gravitational field itself and any matter fields coupled to it.

Here a lattice path integral model of loop quantized four dimensional Euclidean gravity is proposed as a step toward such a theory. The model is a formal quantization of the classical lattice model of [Rei97a], which reproduces, in a continuum limit, Plebanski's form of Euclidean general relativity [Ple77]. The model (and a large class of others like it) can be formulated so that the defining path integrals are sums over "spin worldsheets", the space-time worldsheets of spin networks. (Spin networks are graphs with edges and vertices carrying labels, and are closely related to the loops of loop quantization. Each spin network embedded in space defines a state created by a certain finite polynomial of Wilson loops living on the graph, and together these states span the state space of loop quantization.)

The advantages of a path integral formulation of loop quantization have been recognized for some time [Bae94],[Rei94]. In particular, [Rei94] proposes the representation of quantum gravity as a sum over 4-diffeo equivalence classes of worldsheets of spin networks (spin worldsheets), and also the interpretation of these classes as discrete spacetime geometries. What was needed was a way to translate the dynamics of gravity into this new framework.

Techniques for expressing lattice gauge theories in terms of sums over the worldsheets of electric flux loops go back all the way to Wilson's strong coupling expansions [Wil74]. In 1994 Iwasaki [Iwa94, Iwa95] found a formulation of the Ponzano-Regge model of 2+1 Euclidean GR in terms of a sum over the

whole loop quantization approach to QGR would have to be abandoned, and with it the kinematical prediction of a discrete spectrum of areas.

worldsheets of loops, and the author found a formulation of arbitrary $SU(2)$ lattice gauge theories in terms of lattice spin worldsheets.⁵

Four dimensional Euclidean GR is an $SU(2)$ gauge theory when expressed in terms of self-dual variables, as in Ashtekar's canonical formulation, or Plebanski's covariant formulation [Ple77]. Thus the methods of [Rei94] could be used to translate a suitable lattice model of GR in terms of self dual variables into the spin worldsheet language. However, at the time no such lattice formulation of quantum GR existed.

In 1996 the author presented⁶ a simplicial model in terms of a sum over spins and $SU(2)$ link variables to which the formalism of [Rei94] can be applied directly. This "spin sum" model was obtained by a formal quantization of a classical simplicial action that reproduces Plebanski's formulation of Euclidean GR in a continuum limit [Rei97a].

These developments stimulated Rovelli and the author [RR97] to construct a spin worldsheet sum for transition amplitudes from canonical loop quantized GR, using Thiemann's [Thi96b] proposal for the Hamiltonian constraint. Such an approach has the advantage that one begins immediately in the continuum. However, the quantity they were able to express as a sum over worldsheets, the exponential of the Hamiltonian constraint with constant lapse, is more akin to evolution amplitudes of the gravitational field with respect to a physical clock than to the 4-diffeo invariant transition amplitudes considered here.

Markopolou and Smolin [MS97, Mar97] have proposed a variant of the spin worldsheet formalism which incorporates a Lorentzian causal structure intrinsic to the worldsheet and abandons the topological spacetime as a home for the worldsheets.

Here the original spin sum model, and a hypercubic variant of it, are finally published. Recently, after the work presented here was completed, some similar ideas have been presented in [BaCr97] and [Baez97]. In [BaCr97] Barret and Crane propose a simplicial model in a similar vein to the one given here, with the interesting difference that GR is treated as an $SO(4)$ gauge theory. In [Baez97], by Baez, the proposal that GR be represented by a sum

⁵Independnetly a worldsheet formulation of $U(1)$ gauge theories was found by Aroca, Baig, and Fort [ABF94], and the beginnings of such a formulation for $SU(2)$ theories were developed by Aroca, Fort and Gambini [AFG96].

⁶Seminars, Center for Gravitational Physics and Geometry, Penn State, April '96 and Erwin Shrödinger Institute, Vienna July '96.

over 4-diffeo equivalence classes of spin worldsheets is presented carefully, and related to 2-category theory [Cra95][Bae95].

The present paper consists of two halves, the first dealing with the lattice and spin worldsheet formalism in general, and the second with the particular models for quantum Euclidean GR being proposed.

§2 reviews the formalism of [Rei94] from a somewhat modified perspective. In §2.1 "local" $SU(2)$ lattice gauge theories, which can be represented by spin worldsheet sums, are defined. The procedure for calculating the worldsheet amplitudes for such a lattice model is given in §2.2. §2.3 and §2.4 deal with details omitted in the previous two subsections. The section closes with §2.5 which describes the geometrical interpretation of spin worldsheets.

In §3 a lattice model for four dimensional Euclidean quantum general relativity of the type defined in 2.1 is proposed. §3 presents the simplicial model, and §3.2 motivates this model by showing how it is a formal quantization of the classical simplicial model of [Rei97a]. §3.3 discusses the spin worldsheet formulation of this model.

2 Lattice gravity as a path integral over spin worldsheets

2.1 Local lattice gauge theories of gravity

In the class of lattice models we will consider spacetime is represented by a complex, Π , of four dimensional cells, each having the topology of a 4-ball. Π forms the "lattice", which need not be hypercubic or regular in any way, but which is a piecewise linear manifold.

The physical degrees of freedom in the different cells communicate via boundary data, which is required to match on the mutual boundaries of adjacent cells. Specifically, the boundary data on a cell ν is an $SU(2)$ lattice connection, consisting of $SU(2)$ parallel propagators along the edges of a lattice $[\partial\nu]^*$ on $\partial\nu$, which is dual to $\partial\nu$ seen as a three dimensional cellular complex. I will call $[\partial\nu]^*$ the "dual boundary" of ν . ($[\partial\nu]^*$ is illustrated in Fig. 1). Because the cells communicate only via boundary data I call these models "local". In the lattice gravity models we will consider the $SU(2)$ lattice connection serves as a discrete analog of Ashtekar's $SU(2)$ connection

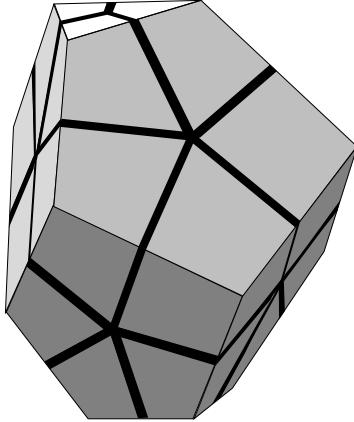


Figure 1: The heavy black lines show the edges of the dual boundary $[\partial\mu]^*$ of a 3-cell μ . In a 4-cell, which is difficult to draw, the boundary is a three dimensional cellular complex, and the edges of the dual boundary connect the centers of the cells of this complex.

for Euclidean GR.⁷

A quantum dynamical model within this framework is characterized by an $SU(2)$ gauge invariant quantum amplitude $a_\nu(\mathbf{g}_{\partial\nu})$ for the connection $\mathbf{g}_{\partial\nu}$ on the cell boundary. This amplitude can be thought of as the Hartle-Hawking state for a spacetime consisting of one cell only. In the path integral formalism it is the exponential of the action for the cell with the given boundary data.

The simplicial model of [Rei97a] provides an action for 4-simplex cells which can be exponentiated to yield such a cellular amplitude. The action of a cell in [Rei97a] in fact depends on several variables in addition to the connection on the boundary. However the connection is the only boundary data, i.e. the only data which is required to match between neighboring cells, so the exponential of the cell action could be integrated over the remaining, internal, variables to yield an amplitude depending only on the boundary connection.

The continuum limit of the classical simplicial model of [Rei97a] is Euclidean GR, in the sense that on sequences of simplicial histories that con-

⁷See [Mar97] for an application of a related formalism to the the “causal” evolution scheme of [MS97].

verge (in a sense defined in [Rei97a]) to continuum gravitational field histories both the simplicial action and field equations converge to those of GR.⁸ ⁹ Using the cell amplitude a_ν obtained from this action ensures that the continuum limit of the naive classical limit of the quantum model is GR. (Here the naive classical limit is the classical theory whose solutions are the stationary points of the quantum amplitude for histories). Of course we really want the continuum classical theory to emerge in a quite different way in quantum theory of gravity. We want it to emerge, at least in some of the states which represent large universes, as the behaviour of expectation values of observables which probe the “classical domain” of gravity, i.e. phenomena at scales much larger than the Plank scale.¹⁰ Nevertheless, the correctness of the continuum limit of the naive classical limit provides motivation for developing quantum models based on the classical model of [Rei97a].

Work is in progress to obtain a tractable expression for the amplitude obtained by simply integrating out the internal (non-boundary) degrees of freedom in the exponentiated classical cell action. However, preliminary results suggest that this amplitude is quite complicated.

Once the model has been specified by the choice of a cell amplitude a_ν , the (unnormalized) amplitude $A(\mathbf{g}_{\partial\Pi})$ for the connection $\mathbf{g}_{\partial\Pi}$ on the boundary of the whole spacetime cellular complex Π is obtained by multiplying together the amplitudes for all the individual 4-cells and then integrating over the connection on the mutual boundaries of the cells, i.e in the interior of Π ,

⁸In [Rei97a] it is emphasized that the Regge model does *not* converge to GR in this sense. I now think this may not reflect a real problem with the Regge model. Solutions of the Regge model may simply converge to continuum GR solutions in a weaker sense than that required in [Rei97a]. Indeed, W. Miller and Gentle [GeMi97] observe this in the case of Kasner cosmologies.

⁹The model of [Rei97a] approximates the Plebanski form of GR [Ple77][CDJM91], which is not couched in terms of the metric and which extends GR to certain degenerate space-times not allowed in the standard metric formulation.

¹⁰In this statement it has been assumed that quantum gravitational corrections to expectation values go to zero as the Plank scale goes to zero relative to all other scales in the problem, and so that the behaviour of the gravitational field is classical at all length scales well above the Plank scale. It has, however, been suggested that, contrary to these *prima facie* reasonable expectations, quantum effects should also be important at cosmological scales [AMM97]. The identification of the classical domain is not trivial.

using the Haar measure to integrate over $SU(2)$ group elements:

$$A(\mathbf{g}_{\partial\Pi}) = \prod_{e \in X} \int dg_e \prod_{\nu \text{ 4-cell of } \Pi} a_\nu, \quad (1)$$

where X is the set of connection bearing edges in the interior of Π , and g_e is the parallel propagator along the edge e .

The connection, being the boundary data, has to match on the mutual boundaries of cells. An important detail is that this makes it necessary to specify a parallel propagator ($SU(2)$ group element) for each of *half* of each edge of $[\partial\nu]^*$, because half edges of $[\partial\nu]^*$, not whole edges, in the boundaries of neighboring cells overlap. (Recall Fig. 1.) More precisely: an edge t of $[\partial\nu]^*$ runs from a vertex inside one 3-cell of $\partial\nu$ to another vertex inside a neighboring 3-cell. The half $t^{(-)}$ is the part of t inside the 3-cell in which t begins, and $t^{(+)}$ is the half in the 3-cell where t ends. In the complex Π neighboring 4-cells share a 3-cell of their boundaries (we will suppose each pair of neighbors shares only one 3-cell), so the half edges of their dual boundaries in that 3-cell coincide. Matching the connection therefore requires the parallel propagators along these half edges to match.¹¹

Clearly the amplitude A that results from the integration is a function of the $SU(2)$ elements on all those dual cell boundary half edges that live in the boundary of Π and thus are not integrated over. These half edges on $\partial\Pi$ together form the dual to $\partial\Pi$, so, just as for a single cell, the boundary data for the whole complex Π , consists of the parallel propagators along the (half) edges of the dual $[\partial\Pi]^*$ to the boundary.

2.2 Spin worldsheet formulations

In this subsection the worldsheet formalism of [Rei94] is reviewed from a different perspective, that takes the cell amplitudes of the last section as the

¹¹Specifying the parallel propagators on the half edges of course gives no more $SU(2)$ gauge invariant information than specifying the parallel propagators on the whole edges. Indeed the gauge invariant content of the requirement that the connection on mutual boundaries match can be expressed entirely in terms of the parallel propagators of entire dual boundary edges. One requires the triviality of the holonomies around certain curves formed from dual boundary edges - specifically, the star shaped curves formed by following, in turn, each dual boundary edge that is incident on a given 2-cell in Π (see Fig. 8 a). This gauge invariant condition implies that the connections on the cell boundaries are gauge equivalent to connections that match on mutual boundaries.

starting point. (This approach takes elements from those of [Iwa94, Iwa95] and [RR97]). The resulting formalism is entirely equivalent to that of [Rei94].

The integral (1) over the connection on the interior of Π that yields the amplitude $A(\mathbf{g}_{\partial\Pi})$ is just a discrete version of a path integral over connections. This path integral may be transformed into a sum over "spin worldsheets", which are the worldsheets of spin networks.

Spin networks (s-nets) are graphs with oriented edges carrying non-zero spins $j \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$ ¹² and vertices, with ordered incident edges, carrying "intertwiners".

Intertwiners are $SU(2)$ invariant tensors.¹³ An intertwiner for a given vertex has, for each incident edge e , an index of spin j_e . That is, the intertwiner is a vector of the tensor product space formed from the representation spaces associated with each of the incoming edges. For trivalent vertices all intertwiners are proportional to the Wigner $3-jm$ symbol $(\begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix})$ (essentially a Clebsch-Gordan coefficient. See e.g. [YLV62]), but for higher valence vertices the space of intertwiners is generally multidimensional.

An s-net embedded in a space with connection defines a spin network *function*: a function of the connection which can be thought of as a generalized Wilson loop. Then, given an embedded spin network Γ , the corresponding spin network function is obtained as follows.

1. To each edge e of Γ , carrying spin j_e , associate the spin j_e representation matrix of the parallel propagator along e . Note that this matrix, $U^{(j_e)m}_n$ has one index living at each end of the edge e . m lives at the beginning and n at the end.
2. Contract the indices of the edge parallel propagators with the corresponding indices of the intertwiners at the vertices at the ends of the edges. If an edge is a closed loop the indices of the parallel propagator are contracted with each other, yielding a Wilson loop.

s-net functions can be written as finite polynomials of Wilson loops and span all such polynomials. For this reason they span the kinematical Hilbert space of loop quantization [RS95, Bae96, Fox95, Rei94, Thi96a, DPR96].

¹²The edges carry spins in $SU(2)$ spin networks. In spin networks of another group, G , the edges carry irreducible representations of G .

¹³They are analogous to δ_{ij} , ϵ_{ijk} and their products, which are invariant tensors of proper $SO(3)$.

On a finite lattice s-net functions span the whole space of gauge invariant distributions of the lattice connection.¹⁴ A linearly independent basis s-net functions can be defined by for each unoriented graph a conventional orientation for each edge, and ordering of incident edges at each vertex, and then choosing a linearly independent basis of intertwiners for each vertex of each graph. The basis elements are then labeled by unoriented graphs, carrying spins on each edge, and the name, or label, of a basis intertwiner at each vertex. If the intertwiner bases are chosen orthonormal¹⁵ and the s-net functions are multiplied by a normalizing factor $\sqrt{2j_e + 1}$ for each edge e ,¹⁶ with spin j_e , then the basis of s-net functions is orthonormal with respect to the natural Ashtekar-Lewandowski inner product [AL94]. On a lattice this inner product on functions of the connection reduces to

$$\langle \chi | \theta \rangle = \prod_{l \text{ edge of lattice}} \int dg_l \chi^* \theta, \quad (2)$$

where the Haar measure is used to integrate over group elements. From here on only orthonormal s-net bases will be considered.

The basis s-net functions are closely analogous to the momentum eigenstates of a particle, and just as momentum can be used instead of position as the boundary data for the motion of a particle, so a basis spin network on the boundary of spacetime can be used in place of a connection as the boundary data for the gravitational field, at least in our class of lattice models.

The amplitudes of basis s-nets on the boundary $\partial\Pi$ are the coefficients of the corresponding s-net functions in the expansion of $A(\mathbf{g}_{\partial\Pi})$. Using the orthonormality of the s-net basis these amplitudes can be obtained as

$$A(\Gamma) = \langle \chi_\Gamma | A \rangle \quad (3)$$

¹⁴Any gauge invariant distribution f on C^∞ functions of the lattice connection has an expansion in terms of s-net functions which converges distributionally to f . That is to say, the series obtained by integrating a test function ϕ against each term in the s-net expansion converges to $f[\phi]$. Distributions that are defined on a larger class of functions of the connection form a subset of distributions on the C^∞ functions, so they also have distributionally convergent expansions in terms of s-net functions. (On distributions see [CH62]).

¹⁵Here “orthonormal” means orthonormal with respect to the inner product $(a, b) = \sum_{m=-j}^j a_m^* b_m$ (m is incremented in integer steps in the sum).

¹⁶A closed loop is treated as a chain of open edges joined at bivalent vertices, which have normalized intertwiner $W^m{}_n = \frac{1}{\sqrt{2j+1}} \delta^m{}_n$. The spin network function of a closed loop is thus just the spin j Wilson loop with no further normalizing factor.

$$= \prod_{t \text{ edge of } [\partial\Pi]^*} \int dg_t \chi_\Gamma^* A. \quad (4)$$

Putting this together with our earlier path integral prescription (1) for $A(\mathbf{g}_{\partial\Pi})$ we see that $A(\Gamma)$ can be calculated by multiplying χ_Γ^* with the cell amplitudes $a_\nu(\mathbf{g}_{\partial\nu})$ of all the cells of Π , and integrating over the connection on *all* of Π , including the boundary $\partial\Pi$.

If we now expand the amplitude, $a_\nu(\mathbf{g}_{\partial\nu})$, of the boundary connection of *each cell*, ν , in s-net basis functions we obtain an expression for the amplitude $A(\Gamma)$ which is both a sum over basis s-nets, and an integral over the connection. It turns out to be quite easy to carry out the integration over connections in each term of the sum.¹⁷ The result is a sum over spin worldsheets S for $A(\Gamma)$:

$$A(\Gamma) = \sum_{S, \partial S = \Gamma} w(S). \quad (5)$$

Let's first define spin worldsheets within the context of a single 4-cell ν . A spin network on $\partial\nu$ which consists of a single loop carrying spin j is spanned by a very simple spin worldsheet, namely a disk in ν carrying spin j . For an arbitrary spin network on $\partial\nu$ (consisting of edges of $[\partial\nu]^*$) a spin worldsheet can always be constructed by taking the spin network and shrinking it continuously to point C_ν in the interior of ν . I will call the point C_ν , which may be freely chosen, the “center” of ν . The 2-surface swept out by the shrinking spin network forms the spin worldsheet, where each patch of worldsheet carries the spin of the edge that swept it out, and each branch line carries the intertwiner label of the vertex that swept it out. (The swept out surface can, and will, be chosen to have no self intersections). The spin worldsheets corresponding to different classes of spin networks on $\partial\nu$ are

¹⁷No reversal of the order of integration and summation is needed, because the spin network expansions of the cell amplitudes $a_\nu(\mathbf{g}_{\partial\nu})$ converge *distributionally*. That is to say, the integral of a function ϕ against a_ν is the sum of the integrals of ϕ against the terms in the expansion of a_ν . Thus, though we might heuristically think of the expansions of the cell amplitudes as pointwise convergent, so that we should sum them up to get the integrand in (4) and then integrate over connections to obtain $A(\Gamma)$, the reverse is in fact true: to get $A(\Gamma)$ we must integrate over connections in each term in the expansion of the integrand of (4) and then sum the resulting integrals. This, correct, ordering of integration before summation is of course precisely the starting point for the spin worldsheet formulation I am about to describe.

illustrated in Figs. 2 and 3. In all cases spin worldsheets in a cell consist of one or more faces with the topology of disks, each carrying its own uniform spin, which are joined along branch lines carrying intertwiner labels.

To each spin network on a cell boundary corresponds a particular cell spin worldsheet (modulo diffeomorphisms in the cell), and, clearly, each such spin worldsheet corresponds to a unique spin network. Thus the sum over cell boundary spin networks in the expression for $A(\Gamma)$ can be interpreted just as well as a sum over the possible assignments of cell spin worldsheets to the cells of Π .

When the integral over connections is carried out it turns out that a non-zero contribution is obtained only from those terms in the sum in which all the cell spin worldsheets together form a continuous surface. Moreover, spins and (if suitable intertwiner bases are used) intertwiner labels on this surface must match across boundaries between cells. At $\partial\Pi$ the spins and intertwiner labels on the surface must match those of the boundary spin network Γ .

I shall call the union of the cell spin worldsheets simply “the spin worldsheet”. This surface will generally have branch lines and self intersection points, preventing it from being a true 2-manifold. (See Fig. 4). However, it can always be decomposed into unbranched components, bounded by edges of Γ or branch lines, at which three or more unbranched components meet. These unbranched components may have transverse intersections at isolated points, including self intersections, but otherwise they are 2-manifolds. The matching conditions across inter cell boundaries and at $\partial\Pi$ show that each unbranched component carries uniform spin, and that those unbranched components that meet the boundary $\partial\Pi$ are bounded there by edges of Γ carrying the same spin. The matching conditions also ensure that on $\partial\Pi$ the intertwiner labels on branch lines match those of the vertices of Γ . They leave open the possibility that the intertwiner labels may change at the centers of cells.¹⁸

Since the integral over connections in the expression for $A(\Gamma)$ is non-zero only for those assignments of basis s-nets to the cell boundaries which correspond to spin worldsheets, $A(\Gamma)$ can be represented as a sum over spin worldsheets, with the weight of each worldsheet given by the value of the

¹⁸In lattice Yang-Mills and BF theories, which can be formulated within the present framework [Rei94], one may choose orthonormal intertwiner bases for the various types of spin network vertices once and for all, and the intertwiner labels will be constant along branch lines.

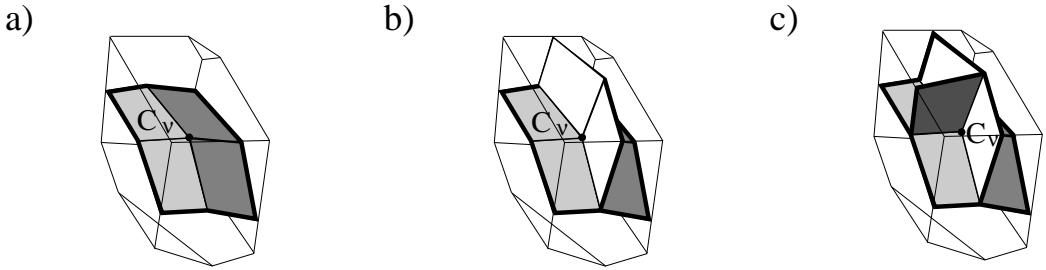


Figure 2: The three panels show the spin worldsheets inside a cell spanning various types of spin networks on the cell boundary. (The spin networks are indicated by heavy lines). Since the four dimensional situation is hard to draw three dimensional analogs are shown.

Panel a) shows spin network consisting of a single unknotted loop of spin j , which is spanned by a disk of spin j .

Panel b) shows a spin network with two vertices and three edges. It is spanned by three faces with the topology of disks which are joined at a trivalent branch line running between the two vertices via the center of the cell. Each face is bounded by one of the edges, and carries the spin of that edge. The two halves of the branch line, on either side of the center, each carry the intertwiner label of the adjacent spin network vertex. These can in general be distinct.

Panel c) shows a spin network with four vertices and six edges. The four branch lines each start at a spin network vertex and end at the center of the cell, which serves as a branch point, or worldsheet vertex.

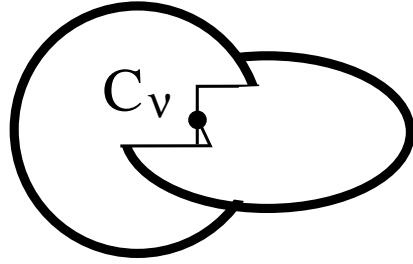


Figure 3: A spin network consisting of two linked loops, and the spin worldsheet that spans it are shown. Since loops cannot be linked in the topological 2-sphere that is the boundary of a 3-cell, I cannot illustrate this type of worldsheet with a three dimensional analog. Instead I have tried to represent the four dimensional situation directly. In depicting knots in a plane one uses breaks in the lines to indicate the parts of the lines that are pushed down below the plane because there is a crossing. Here a broken surface in three dimensions is used to represent a surface in four dimensions, where the breaks indicate regions that are pushed into the fourth dimension, off the 3-space which the viewer is visualizing (see [Car95] for more on this and other techniques of four dimensional visualization). We see that the worldsheet consists of two disks, each spanning a loop, which intersect at a single point - the center of the cell. No attempt has been made to show the 4-cell. The 3-space which the picture images is a 3-surface that cuts through the 4-cell in such a way that it contains most of the spin worldsheet.

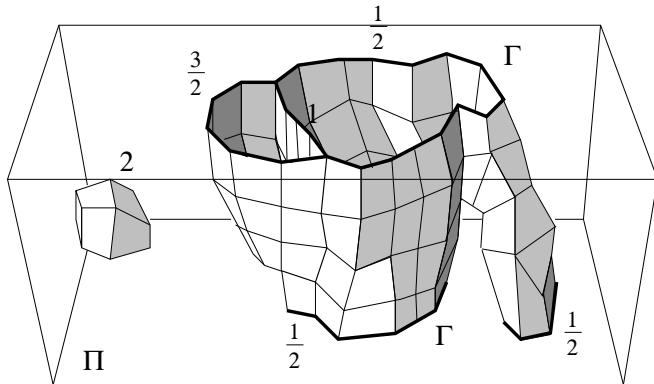


Figure 4: The diagram shows a lattice spin worldsheet in a three dimensional spacetime. The worldsheet consists of four unbranched components, three of which meet along a branch line, while the fourth forms an isolated bubble. The three connected unbranched components also meet the boundary of the spacetime Π on a spin network Γ , which is drawn with heavy lines. A possible assignment of spins to the unbranched components and the edges of Γ have been written in. Note that the branch line carries no intertwiner label since it is only trivalent. To keep the figure simple only the boundary $\partial\Pi$ of the spacetime cellular complex has been shown, and that has been chosen to be a simple box, even though $\partial\Pi$ can, in fact, be quite irregular. The relationship of the spin worldsheet to the individual cells of Π is illustrated in Figs. 2 and 3. The spin worldsheet shown is made up of quadrangles. This is in fact generally true, as will be explained further on in the text. However, the quadrangles will generally not form a rectangular grid. Some possible features of worldsheets that are not illustrated are self intersections at points and non-orientable components.

integral over connections for the corresponding assignment of cell boundary spin networks.

The name spin worldsheet is justified for the surface I have defined because, firstly, it spans the boundary spin network, and secondly, any cross section of the spin worldsheet by a lattice hypersurface, i.e. a 3-surface made of cell boundaries, is a spin network, and thus a legitimate intermediate state in the history of an evolving spin network. The cross sections are spin networks because the edges of a cross section, being each a cross section of an unbranched component, carries constant spin, and because the vertices are vertices of cell boundary spin networks, and thus allowed spin network vertices.¹⁹

That completes the introduction to local lattice gauge theories and their formulation in terms of a path integral over spin worldsheets. To be able to work with these ideas requires a more thorough understanding. In the following two subsections I will return to the main elements of the preceding discussion and develop them more fully. Specifically, I examine the spacetime cellular structure used (§2.3), and the integration over connections that yields the worldsheet sum for $A(\Gamma)$ (§2.4).

2.3 Cellular structure and lattice connections in more detail

Let me begin by describing the cellular complex Π more precisely. Π consists of 4-dimensional cells ν each having the topology of a 4-ball. The boundaries of the 4-cells are made of 3-cells. These either belong to the boundary $\partial\Pi$ of the complex or are shared by precisely two 4-cells. A pair of 4-cells may share at most one 3-cell, and similarly 3-cells may share at most one 2-cell, etc.

In the interior of each d -cell μ we choose a point that we call its “center”, C_μ . Using the centers we will define a cellular substructure of μ (as illustrated in Fig. 5). μ will be divided into “corner cells”, each containing one vertex (or “corner”) of μ . If μ is a 1-cell with endpoints P and Q the corner cell $c_{P\mu}$ is the segment PC_μ . If μ is a 2-cell with a vertex P shared by two 1-cells λ_1

¹⁹For $SU(2)$ spin networks the spins of edges incident at a vertex must satisfy the polygon condition: the spins must be the edge lengths of a polygon of integer circumference which can be realized in the plane [YLV62].

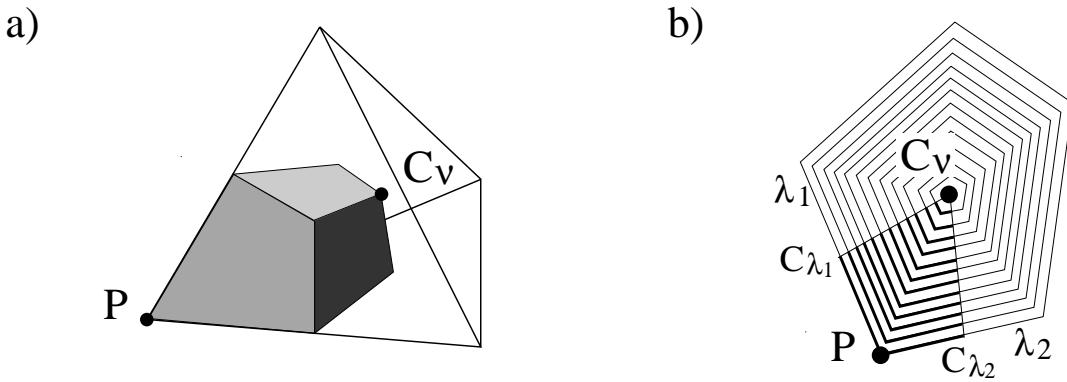


Figure 5: Panel a) shows the corner cell c_P associated with the vertex P of a 3-simplex μ . Notice that the intersection of c_P with any of the triangular faces of μ that are incident on P is itself the two dimensional corner cell of P in the face in question. Note also that c_P is diffeomorphic to a cube, and each of the subsimplices of μ that touch P (including μ and P) contain one corner of c_P . These features are shared by corner cells in any cell (not necessarily a simplex).

Panel b) shows a two dimensional example of the construction of corner cells within a generic polygonal cell μ . The heavy line along the boundary $\partial\mu$ shows the dual boundary cell $P_{\partial\mu}^*$, formed by the union of the two corner cells of P in $\partial\mu$. The images of $P_{\partial\mu}^*$ in the concentric, sucessively smaller images of $\partial\mu$ are also indicated by heavy lines. Together these sweep out the corner cell of P in μ . The 1-cells λ_1 and λ_2 of $\partial\mu$ incident on P , mentioned in the text, and their centers are labeled.

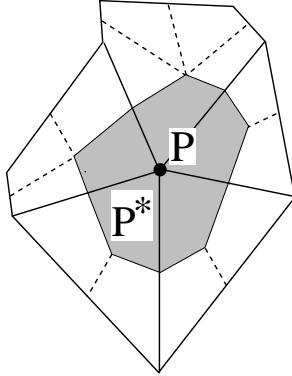


Figure 6: The dual cell P^* dual to the vertex P in a two dimensional cellular complex is shown. The boundaries of the other dual cells are indicated by dashed lines. Note that the cells of the dual complex constructed this way generally are not flat (in a geometry that makes the 4-cells of the original complex flat).

and λ_2 in $\partial\mu$ then $c_{P\mu}$ is the quadrangle $C_\mu C_{\lambda_1} P C_{\lambda_2}$ (See Fig. 5 b.) For μ a cell of arbitrary dimensionality the decomposition into corner cells can be described as follows: each vertex P of μ has a number of $d - 1$ dimensional corner cells in $\partial\mu$. If we let $P_{\partial\mu}^*$ be the union of the corner cells of P in $\partial\mu$, then the corner cell of P in μ is essentially the cone over $P_{\partial\mu}^* \subset \partial\mu$ with vertex at C_μ . More precisely, since $\partial\mu$ is contractible in μ , which has the topology of a ball, one can define a continuous family of diffeomorphic images of $\partial\mu$ that converge on C_μ and cover μ exactly once. Each of these diffeomorphic images contains as a subset the image of $P_{\partial\mu}^*$. The union of these images is the corner cell $c_{P\mu}$. This construction of corner cells is illustrated in Fig. 5 b). Of course the construction only defines the cellular decomposition up to isotopy, but that is all we need.

Notice that the union P_Δ^* of corner cells of the vertex P in the complex Δ forms a cell of a complex Δ^* dual to Δ . See Fig. 6. The dual $[\partial\nu]^*$ of the boundary $\partial\nu$ of a 4-cell ν consists of the 3-cells $P_{\partial\nu}^*$ and the lower dimensional cells derived from these.

So far we have defined three four dimensional cellular complexes, our original cellular complex Π , a complex Π^* which is topologically dual to Π , and a finer complex called the “derived complex”, Π^+ , [Mau96] built of four

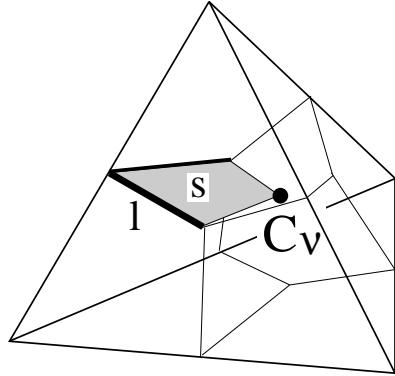


Figure 7: To illustrate the idea of a wedge a wedge of a 3-cell (here a 3-simplex) is shown, though the wedges that are really of interest are wedges of 4-cells. A dual boundary edge l is indicated by a heavy line. Note that this edge starts on a front face of the tetrahedron, but continues on one of the back faces. The other dual boundary edges are indicated by thin lines. The shaded plane region s inside the tetrahedron is the wedge defined by l and the center C_ν of the tetrahedron.

dimensional corner cells, which can be thought of as the intersections of 4-cells of Π and Π^* . From here on a “ d -cells”, “ d^* -cells”, and “ d^+ -cells” will, unless otherwise qualified, be d dimensional cells of Π , Π^* , and Π^+ respectively.

Certain 2-cells of the derived complex, which I will call “wedges”, play a central role both in the building of models and in their spin worldsheet formulation. Fig. 7 shows a wedge in a 3-simplex.

Wedges are those 2-cells of the derived complex which touch the center of some 4-cell of Π . Each wedge is a quadrangle, with one corner at the center of its Π 4-cell ν , and two sides on the boundary of ν . The intersection of the wedge with $\partial\nu$ is in fact an edge of the dual boundary $[\partial\nu]^*$. (See Fig. 7) This follows from the fact that the corner cells are cones, with vertex C_ν , over the 3-cells of the dual boundary.

Wedges are the basic building blocks of spin worldsheets. The cell spin worldsheets discussed earlier (see Figs. 2 and 3) consist of the wedges in the given cell that are bounded by edges of the basis spin network on $\partial\nu$, i.e. by those dual boundary edges that carry non-zero spin. Each wedge carries the spin of its edge.

Finally, it is helpful to note the relationship of the wedges with the dual complex Π^* : The union of all the wedges incident on a 2-cell σ of Π forms the 2-cell σ^* of Π^* which is dual to σ . This is illustrated in Fig. 8.²⁰

It follows from the fact that spin worldsheets can have no open boundaries in the interior of Π that, if a wedge belongs to the worldsheet, then the whole 2*-cell it belongs to must be part of the worldsheet. That is, spin worldsheets are made of entire dual 2-cells. (Note that Π^* has been defined so that the spacetime it covers, i.e. the union of its cells, is the same as that of Π : 2*-cells are therefore cut off beyond $\partial\Pi$.)

A role will be also be played by the edges of Π^+ in the interiors of 4-cells. In a given 4-cell, ν , these edges form the mutual boundaries of the wedges. They connect the center, C_ν , of the 4-cell with the centers of its bounding 3-cells. The centers of the 3-cells are of course the vertices of the dual boundary $[\partial\nu]^*$. Notice also that these edges are halves of 1*-cells. If ν and ν' are two adjacent 4-cells, and τ is the 3-cell forming their mutual boundary, then the edges $C_\nu C_\tau$ and $C_{\nu'} C_\tau$ together form the 1*-cell connecting C_ν and $C_{\nu'}$.

2.4 The integration over connections

Armed with the geometrical imagery of §2.3 we are now ready to tackle the integration over the connection required to find the worldsheet amplitudes in the sum (5) for $A(\Gamma)$. That is, we suppose that the cell amplitudes a_ν in the integrand of (1) have all been expanded on spin network bases, and we now integrate one term in this expansion over connections.

To make the math as clean as possible I will adopt particular conventions regarding the intertwiner bases and the orientations of the wedges. Firstly, I will require that all the wedges in a 2*-cell are coherently oriented. This induces opposed orientations in overlapping pairs of dual boundary edges (see Fig. 8 a). As a result, when we integrate over the parallel propagators along half dual boundary edges the only type of integral that ever appears is

$$\int_{SU(2)} U^{(j_1)}(g)^{m_1}_{n_1} U^{(j_2)}(g^{-1})^{m_2}_{n_2} dg = \frac{1}{2j_1 + 1} \delta_{j_1 j_2} \delta_{n_2}^{m_1} \delta_{n_1}^{m_2}. \quad (6)$$

²⁰The proof is not difficult. One notes that a non-empty intersection $\nu \cap P^*$ of a 4-cell ν and a dual 4-cell P^* is a corner cell. It follows that the portion $\sigma^* \cap \nu$ of a 2*-cell σ^* in ν is a 2-cell in the boundary of a corner cell. Moreover, σ^* must touch C_ν , the only site of Π^* in ν , and σ , the 2-cell it is dual to, so $\sigma^* \cap \nu$ must be the wedge that meets σ .

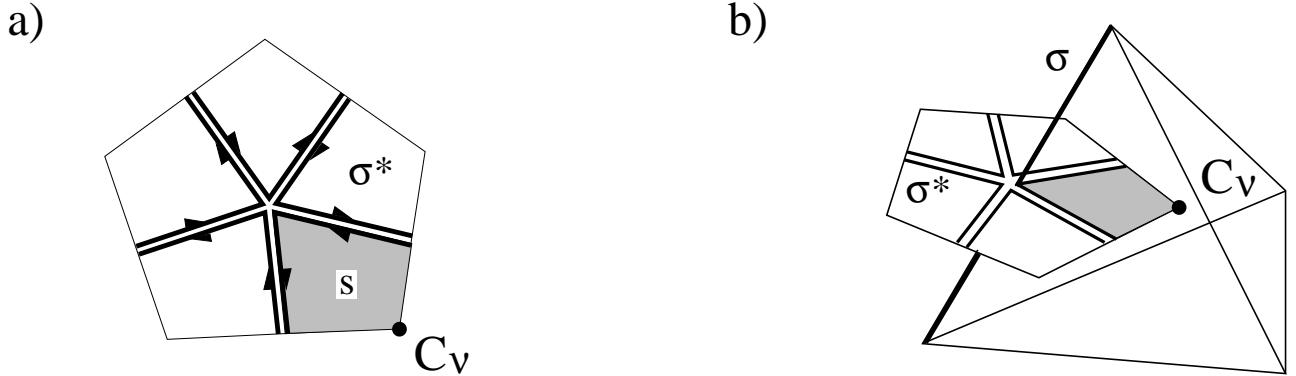


Figure 8: Panel a) shows a 2^* -cell σ^* . σ^* dual to the 2-cell σ of Π , which passes through its center. The corners of σ^* are the centers of the 4-cells of Π incident on σ . The heavy lines show those dual boundary edges belonging to the boundaries of these 4-cells that live on σ^* . σ^* is the union of the wedges associated with these dual boundary edges. In order to write spin network functions on the cell boundaries in terms of parallel propagator matrices we choose orientations for the dual boundary edges. A convenient choice, which will be used, is to choose an orientation for each 2^* -cell, use this to define the orientation of the constituent wedges, and take the consequent orientation of the boundary of the wedge as the orientation of the dual boundary edge that forms part of it. Such a choice of orientation is indicated by arrows on the dual boundary edges in the figure.

Panel b) shows, in the context of a *three* dimensional complex, a 2^* -cell σ^* , the 1-cell σ that it is dual to, and one of the incident 3-cells (a simplex ν).

If ν_1 and ν_2 are two adjacent 4-cells then the edges and the vertices of the dual boundaries $[\partial\nu_1]^*$ and $[\partial\nu_2]^*$ overlap in the 3-cell, τ , shared by ν_1 and ν_2 . The integral (6) arises as the integral over the parallel propagator, g , along a half edge r_1 of $[\partial\nu_1]^*$, lying in this 3-cell. Since the overlapping half edge r_2 of $[\partial\nu_2]^*$ has the opposite orientation it has parallel propagator g^{-1} .

Note that the parallel propagator along an edge, in any representation, is a two point tensor that transforms as a vector under gauge transformations at the beginning of the edge, and as a co-vector under gauge transformations at the end (because it transports a vector from the end of the edge back to the beginning). The upper index m_1 in $U^{(j_1)m_1}_{n_1}$ therefore lives at the beginning of r_1 , while the lower index, n_1 , lives at the end. The indices of $U^{(j_2)}(g^{-1})^{m_2}_{\phantom{(g^{-1})^{m_2}}n_2}$ are similarly housed with respect to r_2 . Since r_2 is antiparallel to r_1 this means that m_2 lives at the end of r_1 , while n_2 lives at the beginning of r_1 . The Kronecker deltas in (6) thus each connect indices living at the same end of r_1 .

The second convention regards the intertwiner bases. I will choose the intertwiner bases at the overlapping vertices of $[\partial\nu_1]^*$ and $[\partial\nu_2]^*$ in τ to be complex conjugates of each other. That is to say, if $\{W_{1I}^{\mathbf{j}}\}$ is a basis of intertwiners for the vertex of $[\partial\nu_1]^*$ (with $\mathbf{j} = [j_1, j_2, \dots]$ the spins of the incident edges at the vertex, and I a label identifying the distinct basis intertwiners for the same \mathbf{j}) then the basis intertwiners of the corresponding vertex of $[\partial\nu_2]^*$ are²¹

$$W_{2I}^{\mathbf{j} m_1 \dots m_a n_1 \dots n_b} = [W_{1I}^{\mathbf{j} m_1 \dots m_a n_1 \dots n_b}]^*. \quad (7)$$

Note that this prescription is compatible with my orientation convention. Complex conjugation turns upstairs indices into downstairs indices, and *vice versa*, so if W_1 is an intertwiner for a vertex v_1 , then W_1^* is an intertwiner for a vertex like v_1 , but with the orientation of all incident edges reversed.

²¹Upstairs indices are vector indices under gauge transformations, and in intertwiners correspond to incoming edges at the vertex, while downstairs indices are covector indices corresponding to outgoing edges of the vertex. Gauge transformations act on co-vectors with the inverse of the vector transformation matrix, so that the contraction of a vector and a co-vector is a scalar. Note that in the unitary representations of the gauge transformations we are using complex conjugation turns vectors into covectors, and *vice versa*, because, being unitary, the gauge transformations preserve the inner product $(a, b) = a^* \cdot b$. It follows that complex conjugation turns upstairs indices into downstairs indices and *vice versa*.

Since we use orthonormal intertwiner bases, in the sense that

$$W_{1I}^{\mathbf{j}} \cdot [W_{1I'}^{\mathbf{j}}]^* = \delta_{II'} \quad (8)$$

(where \cdot signifies contraction on all indices) my convention implies that

$$W_{1I}^{\mathbf{j}} \cdot W_{2I'}^{\mathbf{j}} = \delta_{II'}. \quad (9)$$

That is, the complete contraction of the intertwiners on two overlapping vertices of cell boundary s-nets with the same incident spins is 1 if they carry the same basis intertwiner label I , and zero otherwise.

The integral over connections can now be carried out using (6) and (9). All that needs to be done is to organize the integrations. I will organize them by 2^* -cells. That is, I will do the integrations on all the dual boundary edges living on the same 2^* -cell together. There are two types of 2^* -cells to consider: 2^* -cells in the interior of Π , and 2^* -cells bounded by $\partial\Pi$.

Let's consider a p sided 2^* -cell, σ^* , in the interior of Π . The integral is zero unless all the wedges of σ^* carry the same spin. If the wedges of σ^* do carry a common spin j , then the integration yields

1. a Kronecker delta at each 1^* -cell in $\partial\sigma^*$, i.e. on each 1^* -cell a Kronecker delta on the two dual boundary parallel propagator indices that live there;
2. a chain of contracted Kronecker deltas at the center of σ^* , contributing, when evaluated, a factor $\sum_{m=-j}^j \delta_m^m = 2j + 1$;
3. a factor $(\frac{1}{2j+1})^p$ due to the $\frac{1}{2j+1}$ appearing in the integral (6) and the normalizing factors $\sqrt{2j+1}$ associated with each dual boundary edge.²² $(\frac{1}{2j+1})^p$ can be thought of as a factor of $\frac{1}{\sqrt{2j+1}}$ for each 1^* -cell in $\partial\sigma^*$.

If σ^* is bounded by $\partial\Pi$ the situation is essentially the same. The only difference is that in this case the integral is non-zero only if the wedges *and*

²²Each edge of a basis s-net function carries a normalizing factor $\sqrt{2j+1}$. Because the normalized intertwiner for bivalent vertices is $W_n^m = \frac{1}{\sqrt{2j+1}} \delta_n^m$, the normalization resulting from this convention is unchanged if an edge is split into a chain of edges joined by bivalent vertices.

the edge of the dual boundary $[\partial\Pi^*]$ that bounds σ^* all carry the same spin j . Thus, if $j \neq 0$ σ^* must be bounded by a spin j edge of Γ .

Now let's focus on the vertices of the dual boundaries $[\partial\nu]^*$. These come in overlapping, or "facing", pairs. In the interior of Π a 3-cell contains two overlapping vertices, each belonging to one of the two 4-cells sharing the 3-cell. It is convenient to think of the dual boundary $[\partial\Pi]^*$, on which Γ lives, as a distinct complex that overlaps the dual boundaries of the 4-cells that meet $\partial\Pi$. Then a 3-cell in $\partial\Pi$ contains a vertex belonging to the dual boundary of a 4-cell and a facing vertex belonging to $[\partial\Pi]^*$.

In either case integration over the connection has replaced the parallel propagation matrices that are contracted with the vertex intertwiners in the spin network functions, with Kronecker deltas that contract the intertwiners at facing vertices. (These are the Kronecker deltas at the edges of the 2^* -cells in 1. in the list of factors given above.)

As already explained, the wedges in a 2^* -cell must carry the same spin for the integral over connections to be non-zero. If this is the case the spins of the pair of incident overlapping edges at facing vertices are equal. (9) therefore implies that the contraction of the facing intertwiners is $\delta_{II'}$, where I and I' are the basis intertwiner labels at the two vertices.

Two consequences can be drawn immediately from the above results. Firstly, the fact that only spin assignments in which 2^* -cells carry uniform spin contribute to $A(\Gamma)$ implies that the unbranched components of spin worldsheets consist of entire 2^* -cells. Secondly, the basis intertwiner label on a branch line does not change where the branch line crosses a 4-cell boundary.

To assemble all the factors and obtain an expression for $w(S)$ we now shift our focus to the 4-cells. Three basic types of spin worldsheets can occur inside a 4-cell (recall Fig. 2): a disk; a collection of half disks joined, like the pages of a book, on a branch line crossing the cell; and branch line "vertices", in which three or more branch lines meet at the center of a cell. Since the cell is four dimensional it may also contain several worldsheets of these types, intersecting at the center of the cell. Two disks intersecting at a point are illustrated in Fig. 3. I will refer to both intersections and branch line vertices as vertices of the spin worldsheet.

Let me first show that each unbranched component of a spin worldsheet S carries uniform spin, by considering disk type cell spin worldsheets. Let ν be a 4-cell containing a cell spin worldsheet $S \cap \nu$ consisting of a disk, and possibly other components that intersect the disk at the center of the cell.

Recall that the centers of 4-cells are sites of the dual lattice Π^* . The part of the spin worldsheet inside a 4-cell ν will have the topology of a disk iff C_ν is a point on the interior of an unbranched component.²³ The spin network on $\partial\nu$ which bounds the disk is just a single loop, and thus carries a single uniform spin. This implies that the disk, and thus all the 2^* -cells incident on C_ν , must carry the same spin. Applying this argument to every interior point of an unbranched component then shows that the whole component must carry uniform spin.

Assembling all the factors that contribute to the amplitude of an unbranched component ς of spin j , we find:

1. from the integration over connections, a factor of $\frac{1}{\sqrt{2j+1}}$ for each edge of the boundary of each 2^* -cell in ς . Thus there is a factor of $\frac{1}{2j+1}$ for each interior 1^* -cell of ς , and a factor $\frac{1}{\sqrt{2j+1}}$ for each 1^* -cell in $\partial\varsigma$;
2. also from the integration over connections, a factor of $2j+1$ for each 2^* -cell of ς ;
3. for each 0^* -cell (center of a 4-cell) in the interior of ς the amplitude of the cell boundary spin network $a_\nu[\gamma]$, with $\gamma = \varsigma \cap \partial\nu$.

The amplitude $w(S)$ of the whole spin worldsheet S is then the product of the amplitudes of the unbranched components, times the cell amplitudes a_ν of empty 4-cells, and of ones containing branch lines and vertices.

This result can be put in a more transparent form by working with "reduced amplitudes" for spin networks and spin worldsheets, obtained from $A(\Gamma)$, $a_\nu(\gamma)$, and $w(S)$ by a change of normalization. The reduced amplitude $\bar{A}(\Gamma)$ of the spin network Γ on $\partial\Pi$ is $A(\Gamma)$ multiplied by a factor $\frac{1}{2j+1}$ for each closed loop in Γ , and a factor $\frac{1}{\sqrt{2j+1}}$ for each edge connecting vertices of valence ≥ 3 . The reduced cell amplitude $\bar{a}_\nu(\gamma)$ is defined analogously, and $\bar{w}(S)$ is obtained by multiplying $w(S)$ by the normalizing factor associated with its bounding spin network $S \cap \partial\Pi$. Therefore,

$$\bar{A}(\Gamma) = \sum_{S, \partial S = \Gamma} \bar{w}(S). \quad (10)$$

²³Recall also that in our definition of unbranched components isolated intersection points don't count as branching. So such an intersection point can be an interior point of an unbranched component

When the product of the cell amplitudes in $\bar{w}(S)$ is rewritten in terms of reduced cell amplitudes the factors of $2j + 1$ stemming from the change to reduced amplitudes and those stemming from the integration over the connection combine into, simply, a factor

$$(2j_\varsigma + 1)^{\chi[\varsigma]} \quad (11)$$

for each unbranched component, where $\chi[\varsigma]$ is the Euler characteristic of ς . For any 2-surface of Π^*

$$\chi = N_2 - N_1 + N_0, \quad (12)$$

where N_i is the number of i^* -cells in the surface. As is well known the Euler characteristic of a surface depends only on its topology.

The reduced spin worldsheet amplitude can thus be written as

$$\bar{w}(S) = \prod_{\varsigma \text{ unbranched component}} (2j_\varsigma + 1)^{\chi[\varsigma]} \prod_{\nu \text{ 4-cell of } \Pi} \bar{a}_\nu(\gamma[S]). \quad (13)$$

That is almost the end of the story.²⁴ One detail remains: The reduced cell amplitudes $\bar{a}_\nu(\gamma)$ depend on the orientations of the edges of γ and the ordering of the incident edges at the vertices. In (13) the ordering of incident edges is chosen separately for each facing pair of dual boundary vertices and the orientations of the dual boundary edges are induced by the orientations chosen for the 2^* -cells.

If γ is a single loop the choice of orientations for the 2^* -cells will determine orientations for segments of γ which are generally not coherent, and the sense of circulation defined by the ordering of incident edges at the bivalent vertices will also generally not be coherent. However, the amplitudes for incoherently oriented loops can all be expressed in terms of the amplitude for a coherently oriented loop. Reversing an edge or a bivalent vertex changes the corresponding spin network function by a factor $(-1)^{2j}$.

The spin networks on the boundaries of 4-cells containing branch lines also have obvious "canonical" orientation and ordering conventions. Such a spin network consists of two vertices joined by a collection of edges. The convention requires us to, firstly, pick an ordering of the two vertices and orient each edge from the first vertex to the second, and, secondly, to order the edges the same way at the two vertices. Finally, we may choose the intertwiner bases at the two vertices to be complex conjugates of each other.

²⁴The sign factors found here were missed in [Rei94].

Lets denote the reduced cell amplitudes with these conventions by $\bar{\alpha}_\nu$. For the empty cells we need to fix no conventions. For a 4-cell ν containing a branch line vertex the conventions are fixed by the incident branch lines and unbranched components as follows: Each edge of the spin network $\gamma = S \cap \partial\nu$ is oriented antiparallel to the corresponding (overlapping) edges on the boundaries of the neighboring 4-simplices. At each vertex of γ the ordering of the edges is the same as at the facing vertex on the neighboring 4-cell, and the intertwiner basis is the complex conjugate of that at the facing vertex.²⁵

When $\bar{w}(S)$ is rewritten in terms of the amplitudes α_ν , various sign factors must be kept track of. Organizing these is not quite trivial. I will simply quote the result [Rei97x]: Define the "odd surface" ω to be the surface formed by the union of all the unbranched components carrying odd-half-integer spin. This surface will generally have only even valence branch lines. Those branch lines of S at which only two odd-half-integer spin unbranched components meet will be taken to define bivalent branch lines of the odd surface. Each valence v branch line of ω contributes a factor $(-1)^{\frac{1}{2}v}$, each boundary component contributes a -1 , and finally there is a factor $(-1)^\chi[\omega]$ which is negative for some non-orientable ω .

Our final formula for the reduced spin worldsheet amplitude is therefore

$$\bar{w}(S) = \prod_{\zeta \text{ unbranched component}} (2j_\zeta + 1)^{\chi[\zeta]} \prod_{\nu \text{ 4-cell of } \Pi} \bar{\alpha}_\nu(\gamma[S]). \quad (14)$$

Notice that the prefactors depend only on the topology of S . Only the reduced amplitudes $\bar{\alpha}_\nu$ can contain non homeomorphism invariant information.

A simple model, which can be accomodated in our formalism, is Ooguri's lattice formulation of BF theory [Oog92], which in three dimensions is identical with the Ponzano-Regge model of Euclidean 2+1 GR [PR68].

The cell amplitude for this model can be written as

$$a_{\nu BF}(\mathbf{g}_{\partial\nu}) = \prod_l \int dh_l v_{\nu BF} \quad (15)$$

$$v_{\nu BF} = \prod_{s \text{ wedge of } \nu} \sum_{j_s} (2j_s + 1) \operatorname{tr} U^{(j_s)}(g_{\partial s}). \quad (16)$$

Here the connection has been extended by defining propagators h_l along edges the l of the boundaries of the wedges that connect the center of the 4-

²⁵These conditions do not fix the orientation and other conventions uniquely, but the following results are valid whenever the conditions hold.

simplex with its boundary. The extended connection defines the holonomies $g_{\partial s}$ around the boundaries of the wedges s .

Carrying out the integral over h_l one obtains

$$a_{\nu BF}(\mathbf{g}_{\partial \nu}) = \sum_{\gamma} \chi_{\gamma}^*(\text{flat}) \chi_{\gamma}(\mathbf{g}_{\partial \nu}) \quad (17)$$

where $\{\chi_{\gamma}\}$ is a basis of s-net functions on $[\partial \nu]^*$, and "flat" is a flat connection on $[\partial \nu]^*$. $a_{\nu BF}$ is the flat connection state - integrating any gauge invariant function of the connection against $a_{\nu BF}$ yields the value of that function on flat connections.

From (17) one can read off the reduced amplitudes of cell spin worldsheets:

- For an empty 4-cell $\bar{\alpha}_{\nu} = 1$.
- For a disk $\bar{\alpha}_{\nu} = 1$.
- For a cell spin worldsheet consisting of a segment of branch line $\bar{\alpha}_{\nu} = \delta_{I,I'}$, where I and I' are the intertwiner labels at the vertices where the branch line enters and leaves the cell.
- For a branch line vertex, with cell boundary spin network $\gamma = S \cap \partial \nu$ $\bar{\alpha}_{\nu} = RW[\gamma]$, with $RW[\gamma]$ the Racah-Wigner recoupling coefficient corresponding to the spin network γ .

One sees at once from these expressions, and the general formula (14) for the reduced spin worldsheet amplitude, that the reduced spin worldsheet amplitude is topological, i.e. depends only on homeomorphism invariant features of S .

2.5 Geometrical interpretation of spin worldsheets

Spin worldsheets have a natural interpretation as discrete spacetime geometries [Rei94, RR97]. Working in the context of continuum canonical loop quantized GR Rovelli and Smolin found that the kinematics implies that the spectrum of the observable measuring the area of a given spatial 2-surface σ is discrete. Any spin network state $|\Gamma\rangle$ without vertices on the surface or

points of tangency to the surface is an eigenstate of the area with eigenvalue [RS95][AL96a]

$$\text{area}_\sigma[\Gamma] = \text{Plank area} \sum_{i \in \sigma \cap \Gamma} \sqrt{j_i(j_i + 1)}, \quad (18)$$

where j_i is the spin of the spin network at puncture i .²⁶ Loll [Loll97] has defined an area operator (\mathcal{A}_1 of [Loll97]) in the context of cubic lattice canonical theory with the same eigenvalues.

Even more directly relevant for us is Ponzano and Regge's model of Euclidean 2+1 GR [PR68]. This model is just the BF theory described at the end of §2.4 in the special case of a simplicial, three dimensional spacetime Δ . In Ponzano and Regge's original formulation the connection has been integrated out and the amplitude of a spin network on $\partial\Delta$ is given as a sum over basis spin networks on the boundaries of the 3-simplices, just as in the spin worldsheet formulation. However, they did not view it as a sum over spin worldsheets but rather as a sum over spins on the boundaries of the 3-simplices (Intertwiner labels don't figure because the dual boundaries of 3-simplices have only trivalent vertices, and the space of $SU(2)$ intertwiners for such vertices is one dimensional). Since the spins must match on neighboring cells, there is really only one spin on each 2^* -cell (or, equivalently, on each 1-cell).

Ponzano and Regge noticed that if one defines the length of each 1-cell to be (Plank length) $\times (j + 1/2)$, with j the spin on the 2^* -cell dual to the 1-cell, then in the large spin limit the amplitude for a history in the model is given by

²⁶The action of the area operator on spin network states with vertices in the surface and points of tangency has since been found. However, we will not need this action in our lattice context, since the spin networks we will consider always live on a lattice dual to that of the 2-surfaces whose areas we will evaluate. It might also be worth mentioning that the ambiguity in the spectrum of the area pointed out by Immirzi [Imm96a, Imm96b, Imm96b] exists only in the Lorentzian theory when loop quantized using the Barbero connection [Barb94, Barb95a, Barb95b], and we are doing Euclidean theory.

the exponential of i times the Regge action.²⁷ ²⁸ The Ponzano-Regge model thus approximates the quantum path integral based on the Regge action under those circumstances in which the classical approximation is good, i.e. when 1-cell lengths are much greater than the Planck length. The classical Regge model in turn approximates continuous classical gravitational fields down to a resolution given by the largest 1-cell lengths. Thus the Ponzano-Regge model reproduces classical 2+1 Euclidean GR within its domain of validity, namely in the classical behaviour of modes of wavelength far above the Planck scale, and the Ponzano-Regge state sum can be used as a discrete path integral for Euclidean 2+1 GR.

The lengths of the 1-cells in $\partial\Delta$, the two dimensional "space" of 2+1 GR, are determined, according to Ponzano and Regge's geometrical interpretation, by the spins on the 2^* -cells dual to these 1-cells, or, equivalently, by the spins on the edges of $[\partial\Delta]^*$ dual to the 1-cells. Another way to say this is that the length of a 1-cell in $\partial\Delta$ is determined by the spin on the edge of the boundary spin network that crosses the 1-cell (with the absence of a crossing edge counting as spin $j = 0$). This was noted by Rovelli [Rov93], who pointed out that Ponzano and Regge's definition of edge lengths as $j + \frac{1}{2}$ times the Planck length is equivalent, in the large j limit, to that given by the length operator of two dimensional loop quantization, which gives $\sqrt{j(j+1)} = j + \frac{1}{2} + O(\frac{1}{j})$ in Planck units. The length operator, when applied to the lattice theory, reproduces the geometry of the metric (Regge

²⁷Actually it is approximated by the sum of the exponentials of i times the Regge action evaluated on a set of different geometries having the same edge lengths, but different deficit angles. These alternative geometries are obtained by "folding" the simplicial complex [BaFox94]. Like a sheet of paper being folded, pressed flat, and then glued to a new sheet of paper, the original simplicial complex is mapped continuously (by a many to one mapping) to a new simplicial complex such that the images of some simplices overlap. The deficit angles are then computed in the new simplicial complex.

²⁸The appearance of the i here is peculiar. One normally uses $\exp(-\text{Action})$ as the weight in Euclidean path integrals. The fact that the i appears might mean that the Ponzano-Regge model is not related to the path integral for Lorentzian 2+1 GR as defined by a path integral weighted by $\exp(i\text{Lorentzian Action})$ (although Barret and Foxon [BaFox94] have shown that the Lorentzian classical solutions *do* appear, with weight $\exp(-\text{Lorentzian Action})$, in the Ponzano-Regge state sum). This problem of the i persists in the Euclidean four dimensional models I will discuss, so it may be that such models are not directly connected to Lorentzian GR. In any case a quantization of Euclidean GR with the "wrong complexion", the extra i , would provide a very sophisticated toy model.

calculus) formulation of Euclidean 2+1 GR.

Now let's turn to the case of a four dimensional lattice spacetime Π . On the boundary $\partial\Pi$, where states live, we will adopt the geometry defined by the loop quantized area operator. That is, in a spin network basis state, corresponding to the spin network Γ on $\partial\Pi$ we define the area of a 2-cell in $\partial\Pi$ to be $\sqrt{j(j+1)}$ times the Planck area, with j the spin of the edge of Γ , if any, that punctures the 2-cell (again the absence of such an edge counts as $j=0$).

This geometrical interpretation of spin networks on the boundary can be extended to a geometrical interpretation of spin worldsheets in spacetime. Fix a 2-cell σ in Π . It is always possible to define a three dimensional cellular complex, Σ , containing σ that divides Π into two halves. (For instance, one can take Σ to be the boundary of a 4-cell that σ belongs to). The spin network formed by the intersection of a spin worldsheet S and Σ defines an intermediate state in the history of the gravitational field represented by the spin worldsheet. The area of σ is thus given, via the same expression that applies on $\partial\Pi$, by the spin of the edge of $S \cap \Sigma$ that punctures σ . This is just the spin of S where it punctures σ . The geometrical interpretation of a spin worldsheet, S , is therefore the following [Rei94, RR97]: Each 2-cell σ of Π that is punctured by S has non-zero area, given by

$$\text{area}_\sigma[S] = \text{Planck area} \times \sqrt{j(j+1)}, \quad (19)$$

with j the spin on S where it punctures the 2-cell. 2-cells not punctured by the spin worldsheet have area zero.

In this way the sum over spin worldsheets can be interpreted as a sum over discrete spacetime geometries, which is of course interesting, since the Planck scale discreteness might provide a cure for the ultraviolet divergences of both GR and theories of matter living in the geometrical background established by the gravitational field. Note that the geometrical interpretation of spin worldsheets given here is also well defined for spin worldsheets living in the continuum. (19) then applies to any 2-surface σ puncturing the spin worldsheet only once (transversely at an interior point of an unbranched component). The argument used to derive the geometrical interpretation from the canonical kinematics also works just as well in the continuum.

Another way to state the geometrical interpretation of the spin worldsheet, which shows how very closely analogous it is to Ponzano and Regge's

interpretation of their model is to say that the area of a 2-cell in Π is determined by the spin on its dual 2^* -cell according to (19).

The same type of argument can be used to assign volumes²⁹ to 3-cells punctured by branch lines, which are functions of the intertwiner carried by the branch line at the point of puncture [RR97]. However I will make no use of this possible geometrical interpretation of the intertwiner assignments in the present work.

There is a caveat that the reader should keep in mind. The canonical area operator that has been used as the basis of our geometrical interpretation of spin worldsheets is obtained by quantizing a classical measure of the area by simply substituting operators for the canonical variables in the classical formula. It has *not* been shown that the area of a surface obtained via this operator really reduces to the classical area in a classical limit. Similarly, we can't be *sure* that the spacetime geometry we have defined for worldsheets really reproduces classical spacetime geometry in the classical limit. On the other hand, Geometry is a mathematical construct that we may define any way that is convenient to us, and the geometry we have defined is sufficiently simple and elegant that it may well be a useful concept even if it does not reduce to the classical geometry in the classical limit. Furthermore, it is encouraging that the analogous geometrical interpretation of the Ponzano-Regge state sum does reproduce classical geometry in the classical limit, and I will adopt the geometrical interpretation (19) as a guide in searching for a good dynamical model.

3 A proposal for a model of Euclidean quantum GR

3.1 The model

In this section I propose a specific model for Euclidean quantum GR. It is based on Plebanski's form of GR [Ple77][CDJM91]. More specifically, it is a lattice version of the path integral quantization with histories weighted by $\exp(i \text{ Euclidean action})$. As already pointed out in the context of the

²⁹Volumes are *a priori* independent of the areas since no particular internal geometry of the cells is assumed.

Ponzano-Regge model, such a quantization of the Euclidean theory is rather unconventional, and its relation to the physical Lorentzian theory (in which, at least in the semiclassical sector, the weights of histories should be $\exp(i\text{Lorentzian action})$). Nevertheless Euclidean GR quantized in this way is interesting, at the very least, as a toy model.

Spacetime is represented by an orientable four dimensional simplicial complex Δ . The derived complex Δ^+ , with 4-cells formed by the intersections of the 4-simplices of Δ and the 4-cells of its dual complex Δ^* , plays an essential role in the definition of the model. In particular the wedges are crucial. Recall that a wedge $s(\sigma, \nu)$ is a 2-cell of Δ^+ formed by the intersection of a 4-simplex ν and a 2-cell σ^* of Δ^* dual to a 2-simplex σ in the boundary of ν . (See §2.3 for a discussion of the derived complex, and Fig. 7 for an illustration of a wedge).

In the model a spacetime field configuration (or history) is specified by a collection of spins and $SU(2)$ group elements (parallel propagators). Each wedge s carries a spin $j_s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$. The 1-cells of Δ^+ that bound wedges each carry an $SU(2)$ parallel propagator, defining a lattice connection that specifies the holonomy around the boundary of each wedge.

The amplitude of such a history is

$$w = \prod_{\nu \text{ 4-cell of } \Delta} \text{tr}^{j_1 \otimes \dots \otimes j_{10}} [e^{-\frac{1}{2z^2} \Omega_{ij} \Omega^{ij}} \bigotimes_{s \text{ wedge of } \nu} (2j_s + 1) U^{(j_s)}(g_{\partial s})]. \quad (20)$$

Each 4-simplex ν contributes a factor, a trace in a finite dimensional Hilbert space associated with the 4-simplex. The $g_{\partial s}$ are the holonomies around the boundaries of the wedges s , with base point at the center of ν , and the $U^{(j_s)}(g_{\partial s})$ are the spin j_s representation matrices of these holonomies, each realized on a separate spin j_s representation carrying space associated with to its wedge s .

The trace is taken in the tensor product $\mathcal{H}_{\nu}^{j_1 \otimes \dots \otimes j_{10}}$ of these 10 carrying spaces, as indicated by the superscript $j_1 \otimes \dots \otimes j_{10}$. Finally, $z \in \mathbf{R}$ is a constant adjustable parameter of the model³⁰, and the operator Ω_{ij} is

³⁰Note that if we take $z \rightarrow 0$ then

$$e^{-\frac{1}{2z^2} \Omega_{ij} \Omega^{ij}} = P_{\ker \Omega}, \quad (21)$$

with $P_{\ker \Omega}$ the orthogonal projector onto $\ker \Omega$, the intersection of the kernels of the five independent components of Ω_{ij} . In the original model proposed in [Rei96] this projector

defined as follows: Number the vertices of ν 1,2,3,4,5, so that the vectors 12, 13, 14, 15 form a basis with the same orientation as ν , then

$$\Omega_{ij} = \Gamma_{ij} - \frac{1}{3}\delta_{ij}\Gamma_k^k \quad (22)$$

$$\Gamma_{ij} = \frac{1}{4} \sum_{P,Q,R,S,T \in \{1,2,3,4,5\}} J_{PQRi} \otimes J_{PSTj} \epsilon^{PQRST}. \quad (23)$$

($i, j \in \{1, 2, 3\}$). That is, Ω is the trace free part of Γ . In (23) each oriented 2-simplex σ of ν is represented by the triplet of its vertices ordered in a positive sense around its boundary. J_{PQRi} is the vector of $SU(2)$ generators acting on the holonomy $g_{\partial s(PQR, \nu)}$ around the wedge $s(PQR, \nu)$ associated with the 2-simplex PQR .

Finally, ϵ^{PQRST} is the antisymmetric symbol with $\epsilon^{12345} = 1$. Note that since Ω_{ij} is traceless and symmetric in ij , it is a spin 2 tensor operator acting in $\mathcal{H}_\nu^{j_1 \otimes \dots \otimes j_{10}}$.

Γ_{ij} can be written more compactly as

$$\Gamma_{ij} = \sum_{s, \bar{s} \text{ wedges of } \nu} J_{s i} \otimes J_{\bar{s} j} sgn(s, \bar{s}). \quad (24)$$

For each pair of wedges $(s, \bar{s}) = (s(PQR), s(PST))$ of ν $sgn(s, \bar{s}) = \epsilon^{PQRST}$. $sgn(s, \bar{s})$ can also be defined more geometrically as the sign of the 4-volume spanned by the 2-simplices PQR and PST , i.e. by the ordered set of vectors $\{PQ, PR, PS, PT\}$, with $sgn(s, \bar{s}) = 0$ when the volume is zero. In fact, in any linear coordinate system x^α on ν respecting the orientation of ν ,

$$sgn(s, \bar{s}) = \frac{1}{4!V_\nu} t_{PQR}^{\alpha\beta} t_{PST}^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} \quad (25)$$

$$= \frac{25}{16V_\nu} t_s^{\alpha\beta} t_{\bar{s}}^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta}, \quad (26)$$

where V_ν is the coordinate 4-volume of ν , for any 2-cell c , $t_c^{\alpha\beta} = \int dx^\alpha \wedge dx^\beta$ is the coordinate area bivector of c . sgn is thus simply the translation into simplicial terms of the spacetime antisymmetric tensor density $\epsilon_{\alpha\beta\gamma\delta}$.³¹

appeared in place of the gaussian. However, while $z = 0$ is a natural choice, I will argue that it may lead to difficulties.

³¹The definition of exterior multiplication used here is $[a \wedge b]_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_n} =$

The model defined by (20) fits perfectly into the framework of §2. The cell (4-simplex) amplitude a_ν , as a function of only the connection $\mathbf{g}_{\partial\nu}$ on the cell boundary, is obtained by taking the factor contributed by that cell in (20), and summing over the spins and integrating over the connection in the interior of the cell. Denote by h_l the parallel propagators along the edges l in the boundaries of wedges that connect the center of the 4-simplex with its boundary, then the cell amplitude is

$$a_\nu(\mathbf{g}_{\partial\nu}) = \prod_l \int dh_l \prod_s \sum_{j_s} \text{tr}^{j_1 \otimes \dots \otimes j_{10}} [e^{-\frac{1}{2z^2} \Omega_{ij} \Omega^{ij}} \bigotimes_s (2j_s + 1) U^{(j_s)}(g_{\partial s})]. \quad (27)$$

3.2 Motivation of the model

Why should this model be a quantization of GR? If one replaces the operator $\hat{C} = \exp(-\frac{1}{2z^2} \Omega_{ij} \Omega^{ij})$ with $\mathbf{1}$ in the trace in (20) one obtains Ooguri's simplicial formulation of $SU(2)$ BF theory [Oog92] (see (15)):

$$w_{BF} = \prod_{s \text{ wedge of } \Delta^+} (2j_s + 1) \text{tr} U^{(j_s)}(g_{\partial s}). \quad (28)$$

$SU(2)$ BF theory, which in the continuum formulation has the classical action $\int \Sigma_i \wedge F^i$ ³² with $\Sigma_{i\alpha\beta}$ a triplet of 2-form fields and F^i the curvature of an $SU(2)$ connection, is a topological field theory which is closely related to GR. If one constrains Σ to satisfy the "metricity constraint"

$$\Sigma_i \wedge \Sigma_j - 1/3 \Sigma_k \wedge \Sigma^k = 0 \quad (29)$$

one obtains Plebanski's formulation of full GR [Ple77][CDJ91][Rei95]. When (29) holds and Σ is non-degenerate in the sense that $\Sigma_k \wedge \Sigma^k \neq 0$ one can construct a metric out of the Σ_i . The metrics corresponding in this way to

$a_{[\alpha_1 \dots \alpha_m} b_{\beta_1 \dots \beta_n]}$, where spacetime indices are labeled by lower case greek letters $\{\alpha, \beta, \gamma, \dots\}$. Forms are integrated according to $\int_A a = \int_A \epsilon^{u_1 \dots u_m} a_{u_1 \dots u_m} d^m \sigma$ where A is an m dimensional manifold, σ^u are coordinates on A , the indices u_i run from 1 to m , and $\epsilon^{u_1 \dots u_m}$ is the m dimensional Levi-Civita symbol ($\epsilon^{12\dots m} = 1$ and ϵ is totally antisymmetric).

³²The field Σ is usually called " B ", hence the name "BF theory". Here we use Σ to be consistent with the notation of [CDJ91] for the Plebanski model.

non-degenerate solutions of Plebanski's theory are exactly the solutions to Einsteins field equations.³³

The insertion of the operator \hat{C} in (20), which supresses states in $\mathcal{H}_\nu^{j_1 \otimes \dots \otimes j_{10}}$ with large expectation values of Ω_{ij} in (20) is supposed to be the counter part of the metricity constraint in the classical continuum theory.

To explain this more fully let me make a detour and present the properly Plebanski theory, and also a variant of the classical simplicial model of [Rei97a], that reproduces the Plebanski theory in the continuum limit. The Plebanski theory is defined by the action

$$I_P = \int \Sigma_i \wedge F^i - \frac{1}{2} \phi^{ij} \Sigma_i \wedge \Sigma_j \quad (30)$$

(The euclidean theory is obtained when all fields are real). The lagrange multiplier ϕ^{ij} is a traceless symmetric matrix. The stationarity of the action with respect to this field requires precisely the metricity condition (29). The action is invariant under $SU(2)$ gauge transformations and diffeomorphisms. The indices, which run over $\{1, 2, 3\}$ are spin 1 (adjoint) vector indices under the $SU(2)$ gauge transformations.

On non-degenerate solutions the fields Σ , ϕ , and the $SU(2)$ connection A can be expressed in terms of more conventional variables. Σ is the self-dual part of the vierbein wedged with itself:³⁴

$$\Sigma_i = 2[e \wedge e]^{+0i} \equiv e^0 \wedge e^i + \frac{1}{2} \epsilon_{ijk} e^j \wedge e^k, \quad (31)$$

which transforms as a spin 1 vector under $SU(2)_L$, the left-handed subgroup of the frame rotation group $SO(4) = SU(2)_R \otimes SU(2)_L$, and as a scalar under $SU(2)_R$. A is the self-dual ($SU(2)_L$) part of the spin connection, and ϕ turns out to be the left-handed Weyl curvature spinor.

³³Plebanski's theory also has degenerate solutions, in which the metric is degenerate or altogether undefined, though some geometrical quantites such as areas of surfaces are still defined. It is an extension of standard GR to geometries on which Einsteins field equations are not well defined. Because this extension is not unique (the Samuel-Jacobson-Smolin action [Sam87][JS88] defines a distinct extension)[Rei95], and because the degenerate sector may be important for the quantum theory, I refer Plebanski's formulation of GR as Plebanski's theory.

³⁴The adjoint representation of $SU(2)$ is the fundamental of $SO(3)$, so upstairs and downstairs adjoint representation indices are the same.

Ashtekar's canonical variables are just the purely spatial parts of A and Σ (the dual of the spatial part of Σ is the densitized triad), and, in the non-degenerate sector, the canonical theory derived from (30) is identical to Ashtekar's [CDJM91][Rei95]. Since this is precisely the sector of non-degenerate spatial metric it is of course also equivalent to the ADM theory [ADM62]. However, when the metric is degenerate the canonical theory differs from Ashtekar's [Rei95].

A classical simplicial model, which reduces in the continuum limit to the Plebanski theory, is given in [Rei97a]. We will make use of a slightly modified variant of that model, which has the same continuum limit. The fundamental variables of this model are, in addition to the lattice connection, defined as for the quantum model (20), an $SU(2)$ spin 1 vector $e_{s i}$ associated to each wedge s , which will more or less play the role of Plebanski's Σ_i field,³⁵ and a spin 2 $SU(2)$ tensor φ_ν (represented by a symmetric, traceless matrix, φ_ν^{ij}) associated with each 4-simplex. φ_ν^{ij} plays the role of ϕ^{ij} .

The action for the model is

$$I_\Delta^o = \sum_{\nu \text{ 4-simplex of } \Delta} \left[\sum_{s \text{ wedge of } \nu} e_{s i} \rho_s^i - \frac{1}{60} \varphi_\nu^{ij} \sum_{s, \bar{s} \text{ wedges of } \nu} e_{s i} e_{\bar{s} j} sgn(s, \bar{s}) \right]. \quad (32)$$

ρ_s^i is a measure of the curvature on s . It is a function of the $SU(2)$ parallel propagators via

$$g_{\partial s} = e^{i \rho_s \cdot J} = \cos \frac{|\rho_s|}{2} \mathbf{1} + 2i \sin \frac{|\rho_s|}{2} \hat{\rho}_s \cdot J, \quad (33)$$

where $g_{\partial s}$ the holonomy around ∂s , the boundary of the wedge, $\hat{\rho}$ is the unit vector $\rho/|\rho|$, and the J_i are 1/2 the Pauli sigma matrices³⁶ ρ_s might also be called the "rotation vector" because it is the vectorial angle of the rotation produced by the holonomy $g_{\partial s}$.^{37 38}

³⁵We define $e_{\sigma\nu}$ to reverse sign when the orientation of σ is reversed.

³⁶

$$J_1 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad J_2 = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad J_3 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (34)$$

³⁷Note that ρ_s reverses sign when the orientation of s reverses because the direction of the boundary ∂s reverses.

³⁸ ρ_s^i is singular at $g_{\partial s} = -\mathbf{1}$, but this does not cause any problem in the classical theory.

To recover the Plebanski theory in the continuum limit one must identify the simplicial complex with the spacetime manifold³⁹, the lattice connection with the parallel propagators along the lattice edges computed from the connection A , $e_{s(\sigma,\nu)i}$ with the integral $\int_\sigma \Sigma_i$ ⁴⁰ of Σ_i over the 2-simplex σ , and φ_ν^{ij} with the value of ϕ^{ij} at the center of the 4-simplex ν .

Stationarity of the action (32) with respect to variations of φ_ν^{ij} requires a lattice version of the metricity condition (29) to hold. Namely, it requires

$$\omega_{ij} = \gamma_{ij} - \frac{1}{3}\delta_{ij}\gamma_k^k = 0, \quad (35)$$

where

$$\gamma_{ij} = \sum_{s, \bar{s} \text{ wedges of } \nu} e_{s i} e_{\bar{s} j} sgn(s, \bar{s}). \quad (36)$$

Indeed (35), a non-degeneracy condition, $\gamma_k^k \neq 0$, and the identification of $e_{s(\sigma,\nu)i}$ with the integral of Σ_i over the 2-simplex σ imply that in the continuum limit, and on flat solutions, $e_{s(\sigma,\nu)i}$ is the *metric normal area vector of σ* in an orthonormal reference frame such that σ is purely spatial. (see appendix A of [Rei97a]). This metrical significance of the e_s can also be understood from the point of view of the Ashtekar variables: With the integrands parallel transported to the center of σ along straight lines (according to the linear structure of σ)

$$\int_\sigma \Sigma_i = \int_\sigma \tilde{E}_i^a dn_a \quad (37)$$

where the dreibein density \tilde{E} and the coordinate normal area element dn^a are evaluated in spacetime coordinates for which the cell boundary $\partial\nu$ is an equal time hypersurface.

Now, let's return to the motivation of the quantum model (20). It will not *prove* in any sense that the quantum model (20) really reproduces Euclidean

³⁹The sequence of ever finer simplicial decompositions of the spacetime manifold used to define the continuum limit has to satisfy certain "fatness" conditions ensuring that the piecewise linear structure of the simplicial complexes, and their derived complexes, are compatible with the differentiable structure of the manifold.

⁴⁰The integrand Σ_i is parallel transported from its home on σ , along a straight line (according to the linear structure of ν , first to the center of σ and from there to the center of ν , before the integral is taken. This way the integral is a well defined $SU(2)$ vector living at the center of ν . In [Rei97a] a more elaborate definition of $e_{s i}[\Sigma]$ is used to make proofs cleaner, but the definition given here is adequate.

GR as its classical limit. However, it will be shown how it can be obtained as a formal quantization of the classical simplicial model defined by I_Δ^o .

Since, this classical model has a finite set of degrees of freedom for a finite simplicial complex straightforward path integral methods yield a true quantization of this model: The boundary data for I_Δ^o is the connection. Thus one can define a quantization by setting the amplitude for a given boundary connection to be the integral of $\exp(iI_\Delta^o)$ ⁴¹ over all histories matching the given connection on $\partial\Delta$. This exact path integral quantization has not been completed, but it is sure to lead to a model that is more complicated than (20). (It is noteworthy, however, that preliminary results indicate that the spin 2 tensor operator Ω_{ij} that is so central to (20) also plays a central role in the exact path integral quantization).

The advantage of an exact path integral quantization based on the action I_Δ^o is that its naive classical limit, in which only histories near an extremum of the action contribute to the path integral, reproduces classical Euclidean GR. One would expect the naive classical limit to be realized if the simplicial complex Δ is chosen coarse enough that, according to the classical solution corresponding to the given boundary data, the average 4-volume per cell is much larger than the Planck 4-volume. Then the path integral should be dominated by histories in which most 4-simplices are much larger than the Planck scale, in the sense that their 2-simplices have areas much larger than the Planck area. Such large 4-simplices are each individually in the classical regime, so a semi-classical evaluation of the amplitude for the simplex, dominated by the near extrema of the action, should be accurate. As a result the effective classical action is, modulo small corrections, the "bare" action used to define the path integral.

However, the classical limit of a continuum quantum model defined as some sort of limit of simplicial models as the simplicial complex becomes infinitely fine, then the relation between the cell amplitudes and the classical action is less direct. When Δ is taken fine enough that the classical solution assigns an average 4-volume to the 4-simplices that is much smaller than the Planck 4-volume then one would expect that the path integral is dominated by histories in which the 2-simplices have areas that are of the order of the Planck scale, or zero. Each 4-simplex individually is subject to large quantum

⁴¹The Planck area $a_{\text{Plank}} = G\hbar/c^3$, which divides the action in the exponential, is set to 1.

fluctuations, and there is no particular reason to expect the effective classical action for the complex to be well approximated by the bare action.

Thus the cell amplitude obtained from an exact path integral quantization based on I_Δ^o may not be a good choice for models living on very fine complexes. A better choice as a first guess may be an amplitude, like that defined by (27), that is inspired by the cell amplitude defined by I_Δ^o , but is simpler.

The cell amplitude for the model obtained by path integral quantization with I_Δ^o is

$$a_{\nu o}(\mathbf{g}_{\partial\nu}) = \prod_l \int_{SU(2)} dh_l v_{\nu o}(\{g_{\partial s}\}) \quad (38)$$

$$\begin{aligned} v_{\nu o}(\{g_{\partial s}\}) &= \prod_s \left(\frac{1}{2\pi}\right)^3 \int d^3 e_s \left(\frac{1}{120\pi}\right)^5 \int d^5 \varphi_\nu e^{i \sum_s e_{s i} \rho_s^i - \frac{1}{60} \varphi_\nu^{ij} \sum_{s, \bar{s}} e_{s i} e_{\bar{s} j} sgn(s \bar{s})}, \\ &= \prod_s \left(\frac{1}{2\pi}\right)^3 \int d^3 e_s \delta^5(\omega(e)) e^{i \sum_s e_{s i} \rho_s^i}, \end{aligned} \quad (40)$$

where the $h_l \in SU(2)$ are the parallel propagators along the edges l (of the wedges) connecting the center of ν and the centers of its 3-simplex faces, and δ^5 is the delta distribution on traceless symmetric 3×3 matrices.⁴² (The constant numerical factors in the definition (39) of the unnormalized amplitude $v_{\nu o}$ have been chosen to simplify later results).

$v_{\nu o}$ can be thought of as the wavefunction of a particle in $(\mathbf{R}^3)^{10}$, with coordinates ρ_s^i .⁴³ In particular, it is the state obtained by acting on the wavefunction

$$\Psi = \prod_s \left(\frac{1}{2\pi}\right)^3 \int d^3 e_s e^{i \sum_s e_{s i} \rho_s^i} = \prod_s \delta^3(\rho_s) \quad (41)$$

with the operator

$$\hat{C}_o = \delta^5(\omega(\hat{e})), \quad (42)$$

where $\hat{e}_{s i} = -i \frac{\partial}{\partial \rho_s^i}$ is the momentum conjugate to ρ_s^i .

Now, the delta distribution state Ψ can be written as

$$\Psi = (2\pi^2)^{10} \prod_{s \text{ wedge of } \nu} \sum_{j_s} (2j_s + 1) \operatorname{tr} U^{(j_s)}(g_{\partial s}) \quad (43)$$

⁴² $\delta^5(X) = 2\sqrt{3} \delta(X_{12})\delta(X_{23})\delta(X_{31})\delta(X_{11} - X_{22} + X_{33})\delta(X_{11} - X_{33})$.

⁴³Only ρ_s^i such that $|\rho_s| \leq 2\pi$ correspond to holonomies $g_{\partial s}$.

in the domain $|\rho| \leq 2\pi$. Notice that (aside from the unimportant numerical factor) this is just the amplitude $v_{\nu, BF}$ for the holonomies $g_{\partial s}$ in ν in BF theory, obtained by summing the amplitude for a cell given in (28) over spins. In other words, $v_{\nu, o}$ is obtained by acting on the corresponding BF theory amplitude $v_{\nu, BF}$ with \hat{C}_o which takes out the "non-metrical modes" - the part of $v_{\nu, BF}$ orthogonal to the intersection of the kernels of the operators $\omega_{ij}(\hat{e})$.

The metricity constraint can be softened somewhat by replacing $\delta^5(\omega(\hat{e}))$ by a gaussian

$$\hat{C}_{oz} = e^{-\frac{1}{2z^2}\omega_{ij}\omega^{ij}}. \quad (44)$$

(The factor $\frac{1}{\sqrt{2\pi z}}$ required to normalize the gaussian is absorbed in the overall normalization of the sum over histories).

The amplitude for wedge holonomies, v_{ν} , of (20) is obtained by replacing \hat{e}_{si} by the generator J_{si} in the operator \hat{C}_{oz} and acting with the resulting operator

$$\hat{C} = e^{-\frac{1}{2z^2}\Omega_{ij}\Omega^{ij}} \quad (45)$$

on Ψ .

$\hat{e}_{si} = J_{si}$ at $g_{\partial s} = \mathbf{1}$, so these operators act the same way on Ψ , as do $\omega_{ij}(\hat{e})$ and Ω_{ij} since these are linear in each \hat{e}_s and J_s respectively. Polynomials in \hat{e}_s and J_s act differently since the J_{si} don't commute while the \hat{e}_{si} do. However, in the naive classical limit the path integral is dominated by histories in which the areas of the 2-simplices are much larger than the Planck area. That is to say, $|e_{s(\sigma,\nu)}| \gg 1$. In this limit the difference between low order polynomials $P(\hat{e})$ and $P(J)$ is sub-leading order. If had kept conventional units instead of setting $a_{Planck} = 1$ then the difference would consist of order a_{Planck} terms. Unfortunately this sort of argument does not work for infinite power series like that defining \hat{C} , so there is no proof that $\hat{C}\Psi$ approximates $\hat{C}_{oz}\Psi$.

The model (20) is thus a formal quantization of the classical simplicial model with action I_{Δ}^o .

It is possible to prove that \hat{C} enforces the metricity constraint for nearly continuous histories, that is histories for which the total curvature, ρ_s^i , on each wedge is very small. In this limit

$$\omega_{ij}(\hat{e})\hat{C}\Psi \simeq \Omega_{ij}\hat{C}\Psi. \quad (46)$$

$\Omega_{ij}\hat{C} = \Omega_{ij}e^{-\frac{1}{2z^2}\Omega_{kl}\Omega^{kl}}$ is a hermitian operator with eigenvalues of absolute value $\leq z$. It follows that if z is taken very small violations of metricity will be very small.

Is the substitution $\hat{e} \rightarrow J$ consistent with the geometrical interpretation of spin worldsheets given in §2.5? It seems to be.

Recall that in the classical simplicial model defined by I_Δ^o the area of a 2-simplex σ , in a nearly continuous history which is near a continuum limit solution, is approximately $|e_{s(\sigma,\nu)}|$, where ν is any 4-simplex incident on σ . The field equations ensure that $|e_{s(\sigma,\nu)}|$ is essentially independent of the choice of ν on such histories. In the path integral quantization of this model it is thus consistent with the naive classical limit to take $|e_{s(\sigma,\nu)}|$ as the area of σ estimated within ν , with the true area of σ defined as the common value of the $|e_{s(\sigma,\nu)}|$ when they agree.

Now, recall that $v_{\nu o}(\{g_{\partial s}\}) = \hat{C}\Psi$ is the amplitude for the holonomies $g_{\partial s}$ in ν , obtained by integrating the amplitude $\exp(i \text{ action of } \nu)$ over the e_{si} living in ν . The amplitude for *both* the holonomies in ν and a given value of e_{si} is the component with eigenvalue e_{si} in an expansion of $v_{\nu o}$ into eigenfunctions of the operator \hat{e}_{si} . Similarly, the amplitude for the holonomies and given values of the areas $|e_s|$ in ν is obtained by expanding $v_{\nu o}$ into eigenfunctions of the area operators $\sqrt{\hat{e}_s^2}$.

I will now argue that the expansion

$$\Psi = (2\pi^2)^{10} \prod_{s \text{ wedge of } \nu} \sum_{j_s} (2j_s + 1) \operatorname{tr} U^{(j_s)}(g_{\partial s}) \quad (47)$$

produces essentially such an expansion of $v_{\nu o}$ into area eigenfunctions. $\sqrt{\hat{e}_s^2}$

commutes with \hat{C}_o and⁴⁴

$$\lim_{j_s \rightarrow \infty} \frac{1}{j_s(j_s + 1)} \hat{e}_s^2 \text{tr} U^{(j_s)} = \text{tr} U^{(j_s)}, \quad (52)$$

so the terms $(2\pi^2)^{10} \hat{C}_o \prod_{s \text{ wedge of } \nu} (2j_s + 1) \text{tr} U^{(j_s)}$ are approximate eigenfunctions of the areas with eigenvalues $|e_s| = \sqrt{j_s(j_s + 1)}$ in the limit of large j_s , and thus large areas. It is of course only in this limit, of $|e_s| \gg 1$ = Planck area, that the naive classical limit which justifies our interpretation of $|e_s|$ as area is expected to hold. Identifying $j_s(j_s + 1)$ with the area in Planck units is thus quite consistent with the requirement that the path integral quantization of I_Δ^o reproduces classical GR when the naive classical limit is valid.

The model (20) is obtained by the modification $\hat{C}_o \rightarrow \hat{C}_{oz}$, which is irrelevant for the present discussion, and the substitution $\hat{e}_s \rightarrow J_s$. We shall see in §3.3 that in a spin worldsheet formulation of that model the spins j_s turn out to be exactly the spins of the worldsheets: Each spin distribution that has non-zero amplitude, once the connection is integrated out, is represented by spin worldsheets with the same distribution of spins on the wedges. The substitution $\hat{e}_s \rightarrow J_s$ leads to the geometrical interpretation of spin worldsheets of §2.5.

The reader might wonder how the discrete spectrum of areas defined by the spins arose from the continuous spectrum of $\sqrt{\hat{e}_s^2}$? The key step is (47). The right side provides an expansion of the delta distribution Ψ on the ρ_s in the compact domain $|\rho_s| \leq 2\pi \forall \sin \nu$ corresponding to the product of group

⁴⁴Proof:

$$\hat{e}^2 = -\frac{\partial}{\partial \rho^i} \frac{\partial}{\partial \rho_i} \quad (48)$$

and

$$\hat{e}^2 \text{tr} U^{(j)} = -[\frac{d^2}{d|\rho|^2} + \frac{2}{|\rho|} \frac{d}{d|\rho|}] \frac{\sin(j + \frac{1}{2})|\rho|}{\sin \frac{1}{2}|\rho|} \quad (49)$$

$$= j(j+1) \text{tr} U^{(j)} \quad (50)$$

$$+ (\frac{\cos \frac{1}{2}|\rho|}{\sin \frac{1}{2}|\rho|} - \frac{2}{r}) [(j + \frac{1}{2}) \frac{\cos(j + \frac{1}{2})|\rho|}{\sin \frac{1}{2}|\rho|} - \frac{1}{2} \frac{\sin(j + \frac{1}{2})|\rho| \cos \frac{1}{2}|\rho|}{\sin^2 \frac{1}{2}|\rho|}]. \quad (51)$$

The result then follows trivially when it is noted that the expression in the last line above is bounded on $SU(2)$.

manifolds $SU(2)^{10}$. Instead of requiring the entire continuum of $\sqrt{\hat{e}_s^2}$ eigenfunctions, this requires only a countable set of (approximate) eigenfunctions, with the discrete spectrum of eigenvalues defined by the spins. The situation is quite analogous to what happens when a function is Fourier expanded on a finite interval instead of the whole real line.

3.3 Spin worldsheet formulation of the model

The cell amplitude (27) is very easily expanded on a spin network basis. It is already given as a sum over spins on the wedges, so let's focus on a single term in this expansion, with a particular assignment $\{j_s\}$ of spins to the wedges. Each of the edges $l = C_\nu C_\tau$ connecting the center of the 4-simplex ν with one of its 3-simplex faces τ is shared by four wedges. The h_l dependent part of $\bigotimes_s U^{(j_s)}(g_{\partial s})$ is therefore a direct product of four representation matrices of h_l . Integrating over h_l replaces this product with the projector $\sum_I W_I^* \otimes W_I$, where $\{W_I\}$ is an orthonormal basis of intertwiners for the four valent vertex on τ formed by the wedge boundaries (carrying their respective wedge spins j_s).⁴⁵ The term in the cell amplitude (27) corresponding to the spin assignment $\{j_s\}$ is thus a sum of spin network functions all having the same spins on the edges of $[\partial\nu]^*$, given by the spins of the corresponding wedges, but different intertwiners. If $\{\chi_\gamma\}$ is an orthonormal basis of spin network functions then the whole cell amplitude can be written as

$$a_\nu(\mathbf{g}_{\partial\nu}) = \sum_\gamma \chi(\mathbf{g}_{\partial\nu}) a_\nu(\gamma) \quad (53)$$

$$a_\nu(\gamma) = [\hat{C}\chi_\gamma^*(\mathbf{h})]_{\mathbf{h}=\mathbf{1}}. \quad (54)$$

\mathbf{h} in (54) is a connection on $\partial\nu$. The right side is evaluated on the trivial connection in which every propagator is $\mathbf{1}$.

A spin worldsheet formulation of the model can now be defined using (54) as explained in §2.2. Actually (20) gives the model in a form that is just a step away from a spin worldsheet sum, via the methods of [Rei94]. Each history assigns spins to all the wedges of Δ^+ . This distribution of spins already defines a "spin surface", consisting of the wedges carrying non-zero spin, each one coloured by its spin. These spin surfaces have open

⁴⁵If some of the incident wedges have spin zero then the valence of the vertex is effectively lower.

boundaries inside Δ , they are generally just collections of patches. However, when the connection is integrated out only spin distributions corresponding to spin worldsheets in the sense of §2.2 survive: The contribution of a spin distribution to the amplitude of a spin network Γ on $\partial\Delta$ is

$$\prod_{b \text{ connection carrying edge of } \Delta^+} \int dg_b \chi_\Gamma^* w. \quad (55)$$

with χ_Γ , the spin network function corresponding to Γ , and w the amplitude of the history consisting of the spin distribution and the connection as given by (20). see (4). The only spin distributions for which (55) is non-zero correspond to spin worldsheets spanning Γ . For such a distribution (55) is a sum over spin worldsheets all having the same, given, spins on each wedge, but different intertwiners at the branch lines.

$sgn(s, \bar{s})$ is non-zero iff the pair of wedges (s, \bar{s}) intersects transversely in the sense that they have no common tangent vectors (see (26)). From the expression (54) one sees at once that if the occupied wedges (wedges with $j_s \neq 0$) in a 4-simplex contain no transversely intersecting pair then the reduced amplitudes $\bar{\alpha}_\nu(\gamma)$ are just those of BF theory. That is to say, the reduced amplitude for a cell spin worldsheet consisting of a disk made of wedges none of which intersect transversely is just 1. The reduced amplitude for a branch line through the 4-simplex would be $\delta_{II'}$ where I and I' are the intertwiner labels at the point of entry and exit of the branch line respectively. If we think of BF theory as a "free theory" from which GR is obtained by adding an interaction, then in the spin worldsheet formulation the interaction occurs at the transverse intersections of wedges.

However, in a 4-simplex there are no cell spin worldsheets with branch lines that don't also have transversely intersecting wedges. Similarly, all spin worldsheet vertices have transversely intersecting vertices. The only disks without transversely intersecting wedges are cones made of three wedges. Put another way, in a simplicial complex spin worldsheets have interactions in practically every cell they enter. The only spin worldsheets without interactions are the minimal 2-spheres surrounding 1-simplices, which are the only 2-surfaces that can be made out of cell spin worldsheets that are three wedge disks.

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